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ON THE USE OF SUPERPOSITION IN THE APPROXIMATE SOLUTION OF HEAT CONDUCTION PROBLEMS*

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Abstract—The use of convolution integrals and of the method of images, applied to approximate basic solutions, is investigated. The value of these techniques in deriving approximate solutions of heat conduction problems is discussed, and their accuracy, relative to both known exact solutions and other approximate ones, is assessed.

1. INTRODUCTION

ONE OF the standard techniques for obtaining solutions of many problems in the theory of heat conduction employs the superposition of known simpler solutions. This is principally done in one of two ways: either the solutions which are superimposed pertain to the same geometry but are applied at different times (in which case the process leads in general to a convolution or Duhamel integral [1, 2]), or the superposed solutions are applied simultaneously, at different spatial locations (in which case the procedure is referred to as the method of images [1, 2]). Combinations of the two basic processes just described, as well as other techniques of superposition are at times employed; they all of course rest their validity on the fact that the heat conduction equation with temperature-independent properties is linear, so that sums of solutions are themselves solutions. These methods are known to be extremely powerful in the solution of such linear problems, and their use is correspondingly wide-spread.

The practical value of superposition procedures is high whenever the basic solution employed is already known and is sufficiently simple, since clearly superposition tends to increase the level of complexity of the solution. The question which naturally arises is then, is it possible to simplify matters by employing *approximate* basic solutions in the superposition? Several techniques for the approximate solution of heat conduction problems are well known (for pertinent references and discussions see for example [1, 3, 4]), but to the author's knowledge their use in conjunction with superposition has been neglected in the literature. The aim of the present paper is precisely a study of the feasibility, accuracy and convenience of the use of superposition of approximate basic solutions. Specifically, the following two questions will be investigated here, the reader being referred to [5] for a discussion of other possible extensions of the present approach:

(a) The use of approximate analytical methods of solution for problems with time-dependent applied heating can be obtained either by a convolution integral over a basic (approximate) solution or by direct use of the chosen approximation technique. Which is easier to use, and which more accurate?

(b) In approximate solutions obtained with the aid of the concept of "penetration depth" (i.e. a depth beyond which the surface heating is assumed to have no effect) a new form for

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the solution must be adopted when the penetration depth becomes large enough to reach a new surface point, removed from the region of heat application (i.e. at the "transit time"). The procedure is well known, and clearly requires a new and separate application of the chosen approximation technique after the transit time. The same objective can often be obtained by the techniques of imaging and reflection: again, which procedure is more convenient, and which more accurate?

2. TIME-DEPENDENT BOUNDARY CONDITIONS

We will study here the use of Duhamel, or convolution, integrals on the basis of an approximate solution. To that end, consider the half-space* $x > 0$, initially ($t = 0$) at the temperature $T = 0$, with a prescribed temperature at $x = 0$ given by

$$T(0, t) = T_0(t/t_0)^n \tag{1}$$

where T_0, t_0 and n are constants. Let the approximate temperature $T^{(n)}(x, t)$, corresponding to a particular value of n, be chosen in the form

$$\frac{T^{(n)}(x,t)}{T_0}\left(\frac{t_0}{t}\right)^n = \begin{cases} (1 - x/q_n)^2 & \text{for } x \leqslant q_n(t) \\ 0 & \text{for } x \geqslant q_n(t) \end{cases} \tag{2}$$

where $q_n(t)$ is the penetration depth. If $n = 0$ the solution is well known (for example [3]) and gives

$$q_0 = \sqrt{(A_0\kappa t)};\ A_0 = 147/13,\ 10,\ 12 \tag{3a}$$

respectively for the Biot, Galerkin and heat-balance methods,† where κ is the thermal diffusivity. For any $n \geqslant 0$ we find that ‡

$$q_n = \sqrt{(A_n\kappa t)};\ A_n = \frac{147}{13 + 15n}, \frac{20}{2 + 5n}, \frac{12}{1 + 2n} \tag{3b}$$

respectively for the Biot, Galerkin and heat-balance procedures. This completes the solution, as obtained by direct application of any of the above approximation methods.

We now desire to solve the same problem with the aid the convolution or Duhamel's theorem. The result is [1, 2]:

$$T^{(n)}(x, t) = \int_0^t T^{(0)}(x, t - t_1)\frac{\mathrm{d}}{\mathrm{d}t_1}\left(\frac{t_1}{t_0}\right)^n \mathrm{d}t_1 \tag{4a}$$

$$= nT_0 \int_0^t \left(\frac{t_1}{t_0}\right)^{n-1} \begin{Bmatrix} \left\{1 - \dfrac{x}{\sqrt{[A_0\kappa(t - t_1)]}}\right\}^2 & \text{for } x \leqslant \sqrt{[A_0\kappa(t - t_1)]} \\ 0 & \text{for } x \geqslant \sqrt{[A_0\kappa(t - t_1)]} \end{Bmatrix} \mathrm{d}\left(\frac{t_1}{t_0}\right). \tag{4b}$$

It is shown in [5] that this may be rewritten as

* Or a finite slab before the transit time, see Section 3.

† These three techniques will be used throughout the paper; a description and comparisons among them may be found for example in [3], and in the references cited in [4]. In particular, note that the procedure referred to here as Galerkin's is identical to that which often goes by the name of Kantorovich [6].

‡ The penetration depth is the solution of the equation $q\dot{q}t + nq^2 = 6\kappa t$, in the case of the heat balance method, or

$$q^2 = \frac{12\kappa t}{2n + 1} + \frac{C}{t^{2n}} \quad \text{if } n \neq -\tfrac{1}{2};$$

$$q^2 = 12\kappa t \log Ct \ \text{ if } n = -\tfrac{1}{2}.$$

The constant of integration C is zero for $q(0) = 0$ if $n \geqslant 0$, but is indeterminate if $n < 0$. Similar results are obtained with the other two methods of approximation, since Biot's method gives the equation

$$26q\dot{q}t + 15nq^2 = 147\kappa t,$$

and Galerkin's method gives

$$4q\dot{q}t + 5nq^2 = 20\kappa t.$$

$$\frac{T^{(n)}(x,t)}{T_0}\left(\frac{t_0}{t}\right)^n = \begin{cases} n\displaystyle\int_0^{1-\xi^2} \sigma^{n-1}[1-\xi/\sqrt{(1-\sigma)}]^2\,\mathrm{d}\sigma & \text{for } x \leqslant \sqrt{(A_0\kappa t)} \\ 0 & \text{for } x \geqslant \sqrt{(A_0\kappa t)} \end{cases} \tag{5}$$

where

$$\xi = x/\sqrt{(A_0\kappa t)}. \tag{5a}$$

For $n = 0$ this solution is of course identical with the preceding one, while for example for $n = 1$ and 2 it gives, for $x \leqslant \sqrt{(A_0\kappa t)}$

$$\frac{T^{(1)}}{T_0}\left(\frac{t_0}{t}\right) = 1 - 4\xi + \xi^2(3 - \log\xi^2); \quad \xi \leqslant 1$$

$$\frac{T^{(2)}}{T_0}\left(\frac{t_0}{t}\right)^2 = 1 - \frac{16\xi}{3} + 4\xi^2 + \frac{\xi^4}{3} - 2\xi^2\log\xi^2; \quad \xi \leqslant 1 \tag{6}$$

respectively. This completes the solution by means of the convolution integral.

Finally, the exact solution to the above problem is [1, 2]:

$$\frac{T^{(n)}(x,t)}{T_0}\left(\frac{t_0}{t}\right)^n = 2^{2n}\Gamma(n+1)i^{2n}\operatorname{erfc}\frac{x}{2\sqrt{(\kappa t)}} \tag{7}$$

for n a multiple of 1/2, or [5], for all $n \geqslant 0$:

$$\frac{T^{(n)}(x,t)}{T_0}\left(\frac{t_0}{t}\right)^n = \frac{2}{\Gamma(n+\frac{1}{2})}\int_x^\infty (u-x)^{2n}\,\mathrm{e}^{-u^2}\,\mathrm{d}u. \tag{8}$$

We now wish to compare the direct approximate solution, equation (2) with (3b), with the one given by the convolution integral, equation (5) or (6), and with the exact one, equations (7) or (8).

Note first that all the above approximate solutions are of the penetration-depth form, and that in the convolution case q is independent of n, and is in fact equal to q_0 of the direct solution. In the latter, $q_m < q_n$ for $m > n$, indicating that the higher the power the more restricted the region of significant temperature changes.

Numerical comparisons of the various temperature distributions for $n = 0, 1, 2$ are given* in Fig. 1. They show that the convolution solution is more accurate than the direct one; whether its increased complexity is warranted in any one problem depends on the purpose for which is it needed and must therefore be left to the judgement of the user.

A simpler comparison is obtained by examin-

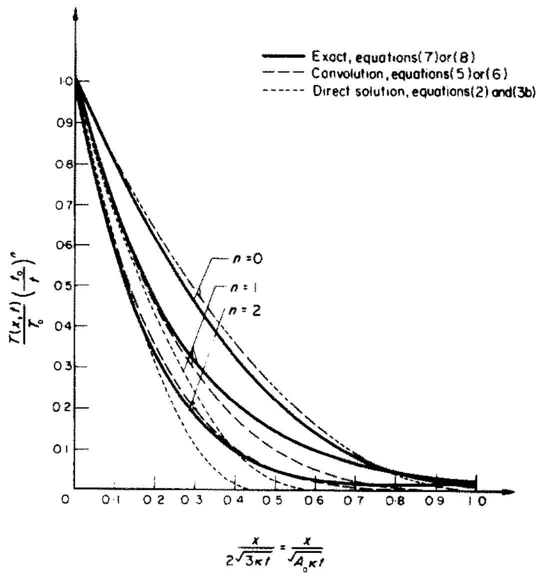

FIG. 1. Temperature distributions in a half-space with time-dependent surface temperature.

* For simplicity, the comparisons in Figs. 1 and 2 are carried out only on the basis of the heat-balance method, although similar results are obtained with the other procedures.

ing the surface heat-input corresponding to the three solutions. Reference [5] shows that the result is:

$$
-\frac{\sqrt{(\kappa t_0)}}{T_0}\left(\frac{t_0}{t}\right)^{n-1/2}\left(\frac{\partial T}{\partial x}\right)_{x=0}
$$

$$
= \begin{cases} \dfrac{2}{\sqrt{A_n}} \text{ by direct solution} & \text{(9a)} \\ 2\sqrt{\left(\dfrac{\pi}{A_0}\right)}\dfrac{n\Gamma(n)}{\Gamma(n+\frac{1}{2})} \text{ by convolution} & \text{(9b)} \\ \dfrac{n\Gamma(n)}{\Gamma(n+\frac{1}{2})} \text{ exact} & \text{(9c)} \end{cases}
$$

The right-hand side of equations (9b) and (9c) differ only by a constant factor, whose value, respectively for Biot's, Galerkin's, and the heat-balance methods, is:

$$
2\sqrt{\left(\frac{\pi}{A_0}\right)} = \sqrt{\left(\frac{26\pi}{147}\right)} \approx 1{\cdot}052;
$$

$$
\sqrt{\left(\frac{2\pi}{5}\right)} \approx 1{\cdot}12; \qquad \sqrt{(\pi/3)} \approx 1{\cdot}023. \qquad (10)
$$

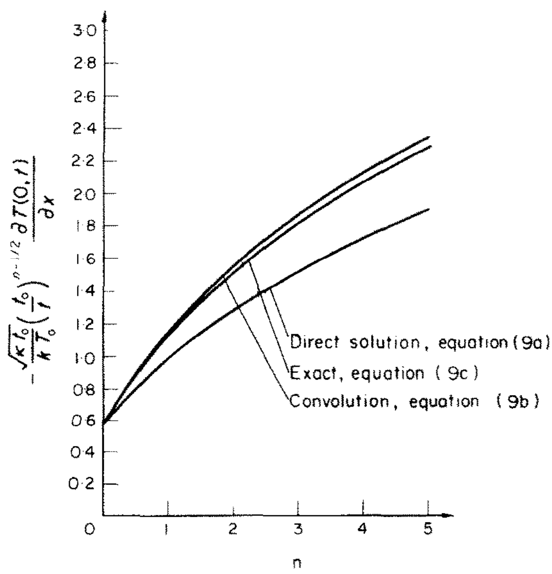

FIG. 2. Surface heat flux in a half-space with time-dependent surface temperature.

Thus the convolution procedure predicts the actual variation with n, with a multiplying factor which does not differ greatly from unity. The direct solution is again simpler but less accurate. A plot corresponding to equations (9a, b, c) is given in Fig. 2. Clearly here also the convolution solution is more accurate than the direct one.

3. IMAGING AND REFLECTION OF APPROXIMATE SOLUTIONS

Consider now a slab of thickness L, initially at zero temperature, whose face $x = 0$ is suddenly raised to a constant temperature T_0 and whose face $x = L$ is insulated.

We will obtain first the solution by direct application of the three methods employed in the preceding work, namely the procedures of Biot, Galerkin and of heat balance. Note first that the solution is

$$
T(x, t) = T^{(0)}(x, t) \text{ if } q_0 = \sqrt{(A_0 \kappa t)} \leqslant L,
$$

$$
\text{i.e. } t \leqslant t_L = \frac{L^2}{A_0 \kappa} \qquad (11)
$$

where $T^{(0)}$ is the half-space solution, equations (2), A_0 is given in equations (3a) for each of the three methods, and t_L is the transit time. For $t \geqslant t_L$, equations (2) are no longer valid, and must be replaced by new expressions, satisfying both the boundary conditions at $x = 0$ and $x = L$, and continuity of temperature distribution at $t = t_L$. The simplest such expression is:

$$
T(x, t) = T_0\left[(1-p)\left(1-\frac{x}{L}\right)^2 + p\right];
$$

$$
p(t_L) = 0; \qquad t \geqslant t_L. \qquad (12)
$$

The parameter $p = p(t)$ is then found to be

$$
p(t) = 1 - \exp\left[-C\left(\frac{t}{t_L} - 1\right)\right] \qquad (13)
$$

where

$$
C = \frac{25}{119} \approx 0{\cdot}218; \tfrac{1}{4}; \tfrac{1}{4} \qquad (13a)
$$

respectively for the Biot, Galerkin and heat-balance procedures. Note that the values of t_L used in (13) differ according to the method used in obtaining C; hence comparisons among the three results are difficult on this basis. A more convenient form is obtained by writing the results in terms of the dimensionless time

$$\tau = \frac{\kappa t}{L^2}. \tag{14}$$

Noting that for the approximate methods $p = T(L, t)/T_0$, we then have

$$\frac{T(L,t)}{T_0} = \begin{cases} 1 - 1{\cdot}2436\,\mathrm{e}^{-2{\cdot}47\tau}; & \tau > 13/147 \approx 0{\cdot}08844 \text{ (Biot)} \\ 1 - 1{\cdot}2840\,\mathrm{e}^{-2{\cdot}5\tau}; & \tau > 1/10 = 0{\cdot}1 \text{ (Galerkin)} \\ 1 - 1{\cdot}2840\,\mathrm{e}^{-3\tau}; & \tau > 1/12 \approx 0{\cdot}0833 \text{ (Heat Balance)} \\ 1 - 1{\cdot}2732\,\mathrm{e}^{-2{\cdot}4674\tau}; & \text{first term of equation (16)} \end{cases} \tag{15}$$

The exact solution can be written in the form [2]:

$$\frac{T(x,t)}{T_0} = 1 - \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{1}{2n+1}\,\mathrm{e}^{-(2n+1)^2\pi^2\tau/4} \sin\frac{(2n+1)\pi x}{2L} \tag{16}$$

in which the infinite series converges rapidly for large times; for example, the first term will give at least 4 correct decimal places in $T(L, t)/T_0$ for $\tau \geqslant 0{\cdot}4$. It is thus evident that direct use of the approximate procedures leads to a "long time" form for the solution, and in this particular problem the Galerkin procedure appears to be the most accurate.

An alternative form of the solution can be derived by imaging the half-space solution, i.e. that denoted in the preceding section by $T^{(0)}(x, t)$. The result is

$$\frac{T(x,t)}{T_0} = T^{(0)}(x,t) + \sum_{n=1}^{\infty}(-1)^n \left[T^{(0)}(2nL + x, t) - T^{(0)}(2nL - x, t)\right]. \tag{17}$$

If the exact expression for $T^{(0)}$ is used, (17) will give an alternative form for the exact solution, or

$$\frac{T(x,t)}{T_0} = \operatorname{erfc}\frac{x}{2\sqrt{(\kappa t)}} + \sum_{n=1}^{\infty}(-1)^n \left(\operatorname{erfc}\frac{2nL + x}{2\sqrt{(\kappa t)}} - \operatorname{erfc}\frac{2nL - x}{2\sqrt{(\kappa t)}}\right) \tag{18}$$

namely the well known short-time form. Thus if approximate forms of $T^{(0)}$ are used in (17), approximate expressions will result which are particularly suited for short times; it is these which we particularly wish to discuss.

Let $T^{(0)}(x, t)$ be chosen in the form of equations (2); then

$$\frac{T^{(0)}(2nL \pm x, t)}{T_0} = \begin{cases} \left(1 - \dfrac{2nL \pm x}{\sqrt{(A_0\kappa t)}}\right)^2 & \text{for } 2nL \pm x \leqslant \sqrt{(A_0\kappa t)} \\ 0 & \text{for } 2nL \pm x \geqslant \sqrt{(A_0\kappa t)}. \end{cases} \tag{19}$$

Care must be exercised to employ the correct value of $T^{(0)}$, according to the two possibilities listed. If this is done, the final result is:

$$\frac{T(L,t)}{T_0} = \begin{cases} 0 & \text{for } 0 \leqslant \tau \leqslant 1/A_0 \\ 2\left(1 - \dfrac{1}{\sqrt{(A_0\tau)}}\right)^2 & \text{for } \dfrac{1}{A_0} \leqslant \tau \leqslant \dfrac{9}{A_0} \\ 2\left(1 - \dfrac{1}{\sqrt{(A_0\tau)}}\right)^2 - 2\left(1 - \dfrac{3}{\sqrt{(A_0\tau)}}\right)^2 & \text{for } \dfrac{9}{A_0} \leqslant \tau \leqslant \dfrac{25}{A_0} \\ 2\left(1 - \dfrac{1}{\sqrt{(A_0\tau)}}\right)^2 - 2\left(1 - \dfrac{3}{\sqrt{(A_0\tau)}}\right)^2 + 2\left(1 - \dfrac{5}{\sqrt{(A_0\tau)}}\right)^2 & \text{for } \dfrac{25}{A_0} \leqslant \tau \leqslant \dfrac{49}{A_0} \end{cases} \tag{20}$$

etc., or, in general

$$\frac{T(L,t)}{T_0} = \frac{(-1)^n 4n}{\sqrt{(A_0\tau)}}\left(1 - \frac{n}{\sqrt{(A_0\tau)}}\right) + [1-(-1)^n]\left(1 - \frac{1}{A_0\tau}\right) \quad (21)$$

$$\text{for } \frac{(2n-1)^2}{A_0} \leqslant \tau \leqslant \frac{(2n+1)^2}{A_0} \qquad n = 0, 1, 2, \ldots$$

where, however, the lower limit of the range of validity is zero if $n = 0$. It is easy to show, for example, that $\lim T(L,t) \to T_0$ as $t \to \infty$ (by letting $\sqrt{(\tau A_0)} = 2n - 1 + \Delta$, $0 \leqslant \Delta < 2$, and taking the limit as $n \to \infty$).

For purposes of comparison with other solutions, let the heat flux at $x = 0$ be calculated. The method of images gives

$$-\frac{L}{2T_0}\frac{\partial T}{\partial x}(0,t) = \frac{(-1)^n}{\sqrt{(A_0\tau)}}\left[1 - \frac{2n+1-(-1)^n}{\sqrt{(A_0\tau)}}\right]; \quad \frac{4n^2}{A_0} \leqslant \tau \leqslant \frac{4(n+1)^2}{A_0} \quad (22)$$

$$n = 0, 1, 2, \ldots$$

Again, this result is particularly accurate for short times, i.e. when the exact solution is approximated by a few terms of the series corresponding to equation (18), or

$$-\frac{L}{2T_0}\frac{\partial T(0,t)}{\partial x} = \frac{1}{2\sqrt{(\pi\tau)}}\left[1 + 2\sum_{n=1}^{\infty}(-1)^n e^{-n^2/\tau}\right]. \quad (23)$$

For longer times, equation (22); although numerically still quite accurate, yields the curious result of a non-monotonic variation of $\partial T(0,t)/\partial x$ with time,* in contrast to the monotonic exact and approximate long-time results; these are, respectively,

$$\frac{-L\partial T(0,t)}{2T_0\partial x} = \begin{cases} \displaystyle\sum_{n=0}^{\infty} e^{-(2n+1)^2\pi^2\tau/4} & \text{from equation (16)} \quad (24a) \\[2ex] 1 - \dfrac{T(L,t)}{T_0} = 1 - p(t) & \text{from equation (12).} \quad (24b) \end{cases}$$

Figure 3 shows a comparison between the exact and approximate results; clearly the solution obtained by imaging is more accurate at short times, while the one obtained by direct use of the approximate procedure is more accurate at long times. For simplicity, this comparison is carried out only on the basis of the heat balance method.

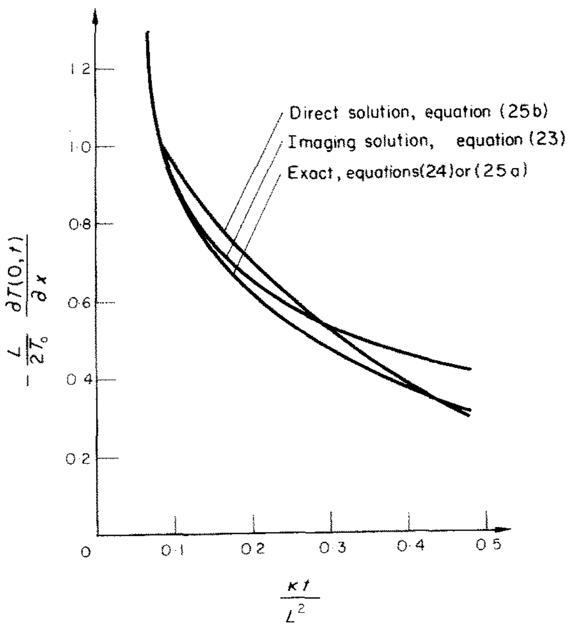

FIG. 3. Surface heat flux in a slab of thickness L.

* In fact, clearly $\partial T(0,t)/\partial x \leqslant 0$ at all times, with the equality sign holding for $\tau A_0 = 16, 64, 144 \ldots$ only.

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SUR L'UTILISATION DE LA SUPERPOSITION DANS LA RESOLUTION APPROCHEE DE PROBLEMES DE CONDUCTION THERMIQUE

Résumé—Les intégrales de convolution et la méthode des images sont appliquées aux solutions approchées fondementales. On discute la valeur de ces techniques en calculant des solutions approchées de problèmes de conduction thermique et on évalue leur précision par rapport aux solutions exactes connues et à d'autres approchées.

DAS SUPERPOSITIONSPRINZIP BEI DER NÄHERUNGSLÖSUNG VON WÄRMELEITUNGSPROBLEMEN

Zusammenfassung—Es wird der Gebrauch von Faltungsintegralen und die Anwendbarkeit der Abbildungsmethoden, zur näherungsweisen Darstellung von Grundlösungen untersucht. Der Wert dieser Methoden zur Herleitung von Näherungslösungen für Wärmeleitprobleme wird diskutiert und die Genauigkeit, relativ, sowohl zu bekannten exakten Lösungen als auch zu anderen Näherungslösungen wird abgeschätzt.

ПРИМЕНЕНИЕ МЕТОДА СУПЕРПОЗИЦИИ ПРИ РЕШЕНИИ ЗАДАЧ ТЕПЛОПРОВОДНОСТИ

Аннотация—Исследуется возможность использования интегралов свертки и метода изображений для приближенных решений фундаментальных уравнений. Обсуждается ценность этого метода для получения приближенных решений задач теплопроводности и проводится оценка их точности по сравнению с известными точными и приближенными решениями.